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A NOTE ON CONJOINT MEASUREMENT WITH RESTRICTED SOLVABILITY

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Abstract

Additive two-factor conjoint measurement is derived from axioms that do not include unrestricted solvability or a condition on interlocked standard sequences.

A NOTE ON CONJOINT MEASUREMENT WITH RESTRICTED SOLVABILITY[1]

This note presents a weakened set of axioms for additive two-factor conjoint measurement. In order to discuss such measurement explicitly, the following notation will be used. The two factors will be denoted by the sets A_1 and A_2; levels of the factors will be denoted by elements a, b, c, ... in A_1, and p, q, r, ... in A_2. Joint effects of combining the two factors will be denoted by pairs of elements, such as ap, bq, ..., in $A_1 \times A_2$. These effects will be assumed to be ordered by a relation $\succsim$ on $A_1 \times A_2$. Unless otherwise specified, all statements about elements such as a and p will be understood to apply to all a in A_1 and p in A_2. In this situation, the representation for additive conjoint measurement states that there are functions ϕ_1 on A_1 and ϕ_2 on A_2, such that $ap \succsim bq$ if and only if $\phi_1(a) + \phi_2(p) \geq \phi_1(b) + \phi_2(q)$. The usual uniqueness result states that the functions are interval scales with the same unit; that is, the functions ϕ_i and ϕ_i' both satisfy the representation if and only if there are constants $\alpha > 0$, β_1, and β_2, such that $\phi_i' = \alpha\phi_i + \beta_i$.

Adams and Fagot (1959) demonstrated that the following three axioms are necessary for the representation.

Axiom 1. $ap \succsim bq$ or $ap \precsim bq$ or both; $ap \succsim bq$ and $bq \succsim cr$ imply $ap \succsim cr$.

Axiom 2. $ap \succsim bp$ implies $aq \succsim bq$; $ap \succsim aq$ implies $bp \succsim bq$.

Axiom 3. $ax \approx fq$ and $fp \approx bx$ imply $ap \approx bq$.

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These axioms have been called respectively weak ordering, independence, and double cancellation.

Luce and Tukey (1964) stated sufficient conditions for conjoint measurement that include two additional axioms, of which one is necessary for the representation and the other is not. The necessary condition is an Archimedean axiom, which will be stated here in the improved version given by Krantz, Luce, Suppes, and Tversky (1971).

Axiom 4. If for some $\bar{a}$, $\underline{a}$ in A_1 and p, q in A_2, a sequence of elements a_i in A_1 satisfies $\underline{a}p \lesssim a_i q < a_i p \approx a_{i+1} q \lesssim a\bar{q}$ for all a_i and a_{i+1}, then the sequence must be finite. A similar statement holds with the roles of A_1 and A_2 reversed.

The nonnecessary condition of Luce and Tukey is a solvability axiom, which requires the sets A_1 and A_2 to be unbounded and is consequently too strong for many empirical applications. Luce (1966) therefore substituted the following restricted solvability axiom.

Axiom 5. For any a, $\bar{b}$, $\underline{b}$ in A_1 and p, q in A_2 such that $\bar{b}q \gtrsim ap \gtrsim \underline{b}q$, there is a b in A_1 such that $ap \approx bq$; for any a, b in A_1 and p, $\bar{q}$, $\underline{q}$ in A_2 such that $b\bar{q} \gtrsim ap \gtrsim b\underline{q}$, there is a q in A_2 such that $ap \approx bq$.

In order to prove conjoint measurement under this weaker solvability condition, Luce added an axiom that assumes the existence of certain interlocked standard sequences. Fortunately, this last rather complicated axiom need not be repeated here, because the present note shows that it can be replaced by the following very weak condition, which Krantz et al. (1971) formulated and called essentialness.

Axiom 6. There are a, b, c in A_1, and p, q, r in A_2, such that $ar > br$ and $pc > qc$.

In other words, an additive conjoint representation, unique up to linear transformations, will be derived from the necessary conditions of weak ordering, independence, double cancellation, and the Archimedean property, and the nonnecessary conditions of restricted solvability and essentialness.

Theorem. If the system $\langle A_1, A_2, \succsim \rangle$ satisfies Axioms 1, 2, 3, 4, 5, and 6, then there are real-valued functions ϕ_i on A_i $(i = 1, 2)$, such that $ap \succsim bq$ if and only if $\phi_1(a) + \phi_2(p) \geq \phi_1(b) + \phi_2(q)$; moreover, the functions ϕ_i' also satisfy the representation if and only if $\phi_i' = \alpha\phi_i + \beta_i$ for some constants $\alpha > 0$, β_1, and β_2.

Proof. By the ordering and independence axioms, the sets A_i can be weakly ordered as follows. For a, b in A_1, let $a \succsim b$ if and only if $ap \succsim bp$ for all p in A_2; for p, q in A_2, let $p \succsim q$ if and only if $ap \succsim aq$ for all a in A_1.

The theorem will first be proved for a square subset $A_1' \times A_2'$, bounded by points $\underline{a} < \bar{a}$ and $\underline{p} < \bar{p}$, such that $\underline{a}\bar{p} \approx \bar{a}\underline{p}$, and $\underline{a}\underline{p} \precsim ap \precsim \bar{a}\bar{p}$ for all ap in $A_1' \times A_2'$. Such a subset must exist by Axioms 5 and 6. For any a in A_1', $\underline{a}\bar{p} \approx \bar{a}\underline{p} \succsim a\underline{p} \succsim \underline{a}\underline{p}$; thus, using Axiom 5, let $\pi(a)$ be such that $\underline{a}\pi(a) \approx a\underline{p}$. Let B be the set of all pairs (a,b) in $A_1' \times A_2'$ such that $a > \underline{a}$, $b > \underline{b}$, and $a\pi(b) \precsim \bar{a}\underline{p}$. For any (a,b) in B, $\bar{a}\underline{p} \succsim a\pi(b) \succsim a\underline{p}$; thus, using Axiom 5 again, let aob be such that $a\pi(b) \approx (aob)\underline{p}$. It will now be shown that if B is nonempty, then the system $\langle A_1', \succsim, B, o \rangle$ satisfies the axioms for bounded extensive measurement as stated by Krantz (1967) or Krantz et al. (1971).

To prove that o is commutative, suppose that (a, b) is in B. By definition, $a\underline{p} \approx \underline{a}\pi(a)$ and $\underline{a}\pi(b) \approx b\underline{p}$; hence, by double cancellation,

$a\pi(b) \approx b\pi(a)$. It follows that (b, a) is in B, and $(aob)\underline{p} \approx (boa)\underline{p}$; thus, by independence, $aob \approx boa$.

To prove that o is monotonic, suppose that (a, c) is in B, and $a \underset{\sim}{>} b$. By independence, $a\pi(c) \underset{\sim}{>} b\pi(c)$. It follows that (b, c) is in B, and $aoc \underset{\sim}{>} boc$. By commutativity, therefore, (c, a) and (c, b) are also in B, and $coa \approx aoc \underset{\sim}{>} boc \approx cob$.

To prove that o is associative, suppose that (a, b) and (aob, c) are in B. Since $aob > b$, it follows that (b, c) is in B. By definition and commutativity, $b\pi(a) \approx (aob)\underline{p}$ and $(boc)\underline{p} \approx b\pi(c)$; hence, by double cancellation, $(boc)\pi(a) \approx (aob)\pi(c)$. By definition and commutativity, $a\pi(boc) \approx [ao(boc)]\ \underline{p} \approx [(boc)\ oa]\underline{p} \approx (boc)\pi(a)$; consequently $a\pi(boc) \approx (aob)\pi(c)$. It follows that (a, boc) is in B, and $ao(boc) \approx (aob)oc$.

The remaining conditions for extensive measurement can be verified immediately. Therefore, if B is nonempty, there is a function ϕ_1 on A_1', such that $\phi_1(aob) = \phi_1(a) + \phi_1(b)$, and $\phi_1(a) \geq \phi_1(b)$ if and only if $a \underset{\sim}{>} b$. For any p in A_2, $\bar{a}\underline{p} \approx \underline{a}\bar{p} \underset{\sim}{>} \underline{a}\underline{p} \underset{\sim}{>} \underline{\underline{a}}\underline{p}$; thus, let $\alpha(p)$ be such that $\alpha(p)\underline{p} \approx \underline{a}\underline{p}$, and let $\phi_2(p) = \phi_1[\alpha(p)]$. By the definitions of $\alpha(p)$ and $\pi(a)$, $\underline{a}\underline{p} \approx \alpha(p)\underline{p} \approx a\pi[\alpha(p)]$; hence, $p \approx \pi[\alpha(p)]$. Thus, for $ap \underset{\sim}{<} \bar{a}\underline{p}$, it follows that $[a, \alpha(p)]$ is in B, and $ap \approx a\pi[\alpha(p)] \approx [ao\alpha(p)]\underline{p}$. Now, suppose that $ap \underset{\sim}{<} \bar{a}\underline{p}$ and $bq \underset{\sim}{<} \bar{a}\underline{p}$. In this case, $ap \underset{\sim}{\geq} bq$ if and only if $ao\alpha(p) \underset{\sim}{\geq} bo\alpha(q)$, which by extensive measurement occurs if and only if $\phi_1(a) + \phi_1[a(p)] \geq \phi_1(b) + \phi_1[a(q)]$, which by definition holds if and only if $\phi_1(a) + \phi_2(p) \geq \phi_1(b) + \phi_2(q)$. In other words, ϕ_1 and ϕ_2 provide a conjoint representation for all $ap \lesssim \bar{a}\underline{p}$. The uniqueness of these scales follows from the uniqueness of extensive measurement.

To show that the representation also holds for the rest of $A_1' \times A_2'$ when B is nonempty, suppose first that $ap \succsim \bar{a}\underline{p}$ and $bq \succsim \bar{a}\underline{p}$. If $a \succsim b$ and $p \succsim q$, or if $a \precsim b$ and $p \precsim q$, the proof is obvious. Suppose, therefore, that $a \precsim b$ and $p \succsim q$; the proof is similar if $a \succsim b$ and $p \precsim q$. Since $b\bar{p} \succsim \underline{a}\bar{p} \approx \bar{a}\underline{p} \succsim b\underline{p}$ and $\bar{a}p \succsim \bar{a}\underline{p} \approx \underline{a}\bar{p} \succsim \underline{a}p$, let r and c be defined such that $br \approx cp \approx \underline{a}\bar{p}$. It follows from double cancellation that $ap \approx bq$ if and only if $ar \approx cq$. It also follows from independence that $ar \precsim br \approx \underline{a}\bar{p}$ and $cq \precsim cp \approx \underline{a}\bar{p}$; hence, by the representation already established, $\phi_1(b) + \phi_2(r) = \phi_1(c) + \phi_2(p)$, and $ar \approx cq$ if and only if $\phi_1(a) + \phi_2(r) = \phi_1(c) + \phi_2(q)$. These facts can be combined to show that $ap \approx bq$ if and only if $\phi_1(a) + \phi_2(p) = \phi_1(b) + \phi_2(q)$. If $ap \succ bq$, then because $bq \succsim \bar{a}\underline{p} \approx \underline{a}\bar{p} \succsim \underline{a}p$ by hypothesis, let d be defined such that $dp \approx bq$, and thus $a \succ d$ by independence; it follows from the definition of ϕ_1 and the last sentence that $\phi_1(a) + \phi_2(p) > \phi_1(d) + \phi_2(p) = \phi_1(b) + \phi_2(q)$. If $ap \prec bq$, a similar argument gives the reversed inequality. These results mean that $ap \succsim bq$ if and only if $\phi_1(a) + \phi_2(p) \geq \phi_1(b) + \phi_2(q)$, as claimed. In the remaining case, $ap \succsim \underline{a}\bar{p} \succsim bq$ if and only if $\phi_1(a) + \phi_2(p) \geq \phi_1(\underline{a}) + \phi_2(\bar{p}) \geq \phi_1(b) + \phi_2(q)$, by the results already proved.

If B is empty, then $\underline{a}$ and $\bar{a}$ must be the only elements in A_1', and it follows from Axiom 5 that $\underline{p}$ and $\bar{p}$ are the only elements in A_2'. By definition, $\underline{a}\underline{p} \prec \underline{a}\bar{p} \approx \bar{a}\underline{p} \prec \bar{a}\bar{p}$. Therefore, for any constants $\alpha > 0$, β_1, and β_2, let $\phi_1(\underline{a}) = \beta_1$, $\phi_1(\bar{a}) = \beta_1 + \alpha$, $\phi_2(p) = \beta_2$, and $\phi_2(\bar{p}) = \beta_2 + \alpha$. These functions clearly satisfy the representation and uniqueness for conjoint measurement.

Next, the representation will be generalized to the hypothesis that $A_1' \times A_2'$ is rectangular. It will be assumed that $\bar{a}\underline{p} \succ \underline{a}\bar{p}$; the proof is

similar in the other case. Let a_1 be such that $a_1\underline{p} \approx \bar{a}\bar{p}$; if a_n is defined and $a_n\bar{p} \lesssim \bar{a}\underline{p}$, let a_{n+1} be defined such that $a_{n+1}\underline{p} \approx a_n\bar{p}$. It has already been shown possible to define conjoint scales ϕ_1 and ϕ_2 on $\underline{a} \lesssim a \lesssim a_1$ and $\underline{p} \lesssim p \lesssim \bar{p}$ respectively; for convenience, let $\phi_1(\underline{a}) = \phi_2(\underline{p}) = 0$ and $\phi_1(a_1) = \phi_2(\bar{p}) = 1$. The scale ϕ_1 can now be extended to the rest of A_1' by induction on n, since by the Archimedean axiom, there must be some m such that $a_m\bar{p} \gtrsim \bar{a}\underline{p}$. Therefore, to perform the induction, it is assumed that $\phi_1(a)$ has been defined with the appropriate properties for all $a \lesssim a_n$.

Suppose that a_{n+1} is defined; the proof is similar if $a_n\bar{p} > \bar{a}\underline{p}$. For any a such that $a_n \lesssim a \lesssim a_{n+1}$, it follows that $a_{n-1}\bar{p} \approx a_n\underline{p} \lesssim ap \lesssim a_{n+1}\underline{p} \approx a_n\underline{p}$; thus, let a' be such that $a'\bar{p} \approx a\underline{p}$. Since $a' \lesssim a_n$ by independence, $\phi_1(a')$ is defined by the induction hypothesis; therefore, let $\phi_1(a) = \phi_1(a') + 1$. For any $a_n \lesssim a \lesssim a_{n+1}$ and p in A_2', $a'\underline{p} \lesssim a'p \lesssim a'\bar{p} \approx a\underline{p}$; thus, let $\delta(ap)$ be such that $\delta(ap)\underline{p} \approx a'p$. The definitions of a' and $\delta(ap)$ imply by double cancellation that $\delta(ap)\bar{p} \approx ap$. To compare ap and bq, suppose first that $a_n \lesssim a \lesssim a_{n+1}$ and $a_n \lesssim b \lesssim a_{n+1}$. Consequently, $ap \gtrsim bq$ if and only if $\delta(ap)\bar{p} \gtrsim \delta(bq)\bar{p}$, which by independence is equivalent to $\delta(ap)\underline{p} \gtrsim \delta(bq)\underline{p}$, which by definition holds if and only if $a'p \gtrsim b'q$, which by the induction hypothesis occurs if and only if $\phi_1(a') + \phi_2(p) \geq \phi_1(b') + \phi_2(q)$, which by definition is equivalent to $\phi_1(a) + \phi_2(p) \geq \phi_1(b) + \phi_2(q)$. Next, suppose that $a_n \lesssim a \lesssim a_{n+1}$ and $b \lesssim a_n$. If $ap \gtrsim a_n\bar{p}$, then $ap \gtrsim bq$, and also $\phi_1(a) + \phi_2(p) \geq \phi_1(a_n) + \phi_2(p) \geq \phi_1(b) + \phi_2(q)$. If $ap \lesssim a_n\bar{p}$, then because $a_n\underline{p} \lesssim ap$, let p' be such that $ap \approx a_np'$; hence, $ap \approx a_np' \gtrsim bq$ if and only if $\phi_1(a) + \phi_2(p) = \phi_1(a_n) + \phi_2(p') \geq \phi_1(b) + \phi_2(q)$. Since a' is defined uniquely

(up to $\approx$) for each a, the uniqueness of $\Phi_1(a)$ is the same as the uniqueness of $\Phi_1(a')$, which is specified by the induction hypothesis.

Finally, the representation can be extended to all $A_1 \times A_2$ by defining a series of rectangular subsets. To fix the upper boundaries of these rectangles, an infinite sequence of points $\bar{a}_n$ will be defined, such that $\bar{a}_{n+1} \succsim \bar{a}_n$ for all n, and given any a in A_1, there is an m such that $a_m \succsim a$. If A_1 is bounded by an $\bar{a}$ such that $\bar{a} \succsim a$ for all a, then let $\bar{a}_n = \bar{a}$ for all n; clearly this sequence has the required properties. Next, suppose that there is no such upper bound. Let three points $\bar{a}_o$, q_o, and q be given, with $q > q_o$. The points $\bar{a}_{n+1}$ and q_{n+1} will be defined recursively, given $\bar{a}_n$ and q_n, with $q > q_n$. If there is an a such that $aq_n \succsim \bar{a}_n q$, then let $\bar{a}_{n+1}$ be such that $\bar{a}_{n+1}q_n \approx \bar{a}_n q$; also, let $q_{n+1} = q_n$. Otherwise, let $\bar{a}_{n+1}$ be any point such that $\bar{a}_{n+1} > \bar{a}_n$; since in this case $\bar{a}_{n+1}q_n < \bar{a}_n q < \bar{a}_{n+1}q$, let q_{n+1} be such that $\bar{a}_{n+1}q_{n+1} \approx \bar{a}_n q$. These definitions immediately imply that for all n, $\bar{a}_{n+1} \succsim \bar{a}_n$ and $q > q_{n+1} \succsim q_n$. Now, let any a in A_1 be given; by the hypothesis of unboundedness, there must also be a b such that $b > a$. By the Archimedean axiom, the portion of the $\bar{a}_n$ such that $\bar{a}_n < a$ and $bq_n < aq$ must be finite; thus, there is an m such that either $\bar{a}_m \succsim a$, or else $a_m < a$ and $bq_m \succsim aq$. In the second of these cases, the portion of the a_n such that $n \geq m$, $a_n < a$, and $q_n = q_m$ must again be finite; thus, there must be an ℓ such that $a_\ell < a$ and $aq_\ell < a_\ell q$. In this case, however, $bq_\ell = bq_m \succsim aq > a_\ell q$; hence, $a_{\ell+1}q_\ell \approx a_\ell q > aq_\ell$, which means that $a_{\ell+1} > a$, as claimed.

In a similar manner, it is possible to define sequences $\underset{\sim}{a}_n$, $\bar{p}_n$, and $\underset{\sim}{p}_n$, such that if the set $A_1^{(n)}$ is defined to contain all a in A_1 with $\underset{\sim}{a}_n \lesssim a \lesssim \bar{a}_n$ and the set $A_2^{(n)}$ is defined to contain all p in A_2 with $\underset{\sim}{p}_n \lesssim p \lesssim \bar{p}_n$, then $A_1^{(n)} \times A_2^{(n)} \subseteq A_1^{(n+1)} \times A_2^{(n+1)}$ for all n, and for any ap in $A_1 \times A_2$, there is an m such that ap is in $A_1^{(m)} \times A_2^{(m)}$. As has already been shown, it is possible for any n to construct functions $\phi_1^{(n)}$ on $A_1^{(n)}$ and $\phi_2^{(n)}$ on $A_2^{(n)}$ with the properties required for conjoint measurement. Moreover, the arbitrary constants in these functions can be set so that for any n, $\phi_1^{(n+1)}(a) = \phi_1^{(n)}(a)$ for all a in $A_1^{(n)}$, and $\phi_2^{(n+1)}(p) = \phi_2^{(n)}(p)$ for all p in $A_2^{(n)}$. Thus, for any a in A, $\phi_1(a)$ can be defined as equal to $\phi_1^{(n)}(a)$, where n is such that a is in $A_1^{(n)}$; similarly, for any p in A_2, $\phi_2(p)$ can be defined as equal to $\phi_2^{(n)}(p)$, where n is such that p is in $A_2^{(n)}$. QED.

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